On Point
AERONAUTICS

Minimizing the Rate of Descent in Circling Flight

Design Parameters for a Fixed-Span Glider

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List of Symbols

A	Aspect ratio
A_1, B_1, B_2	Constants
b	Wingspan
c_{AVG}	Average cord
C_D	Drag coefficient
C_{D_0}	Drag coefficient at zero lift
C_L	Lift coefficient
D	Drag
e	Span efficiency factor
g	Acceleration of gravity
$\dot{h}$	Rate of change in altitude
$\dot{h}_0$	Rate of change in altitude for straight flight
$\dot{h}_\phi$	Rate of change in altitude for circling flight
I_w	Wing weight index
L	Lift
q	Dynamic pressure ($\frac{1}{2}\rho V^2$)
R	Radius of turn
S	Wing area
t/c	Section thickness ratio
V	Velocity (airspeed)
W	Aircraft weight
W_w	Wing weight
γ	Glide path angle
π	Pi
ρ	Air density
ϕ	Bank angle

Introduction

One of the most important points to consider when designing a new glider is the ability to loiter and climb in thermal lift. The flight condition that maximizes this performance is where the rate of descent is a minimum. The lower the rate of descent for a glider design, the longer the glider can stay aloft.

When designing a new glider, it is often the case that the wingspan must be limited to a target value. Sometimes this is because of the width of a hanger or the length of a trailer. Sometimes this is because of manufacturing limits on the wing spar. In the case of competition gliders, they are categorized by their span and a 15-meter span is called the "Standard Class."

It is well known that gliders with a high aspect ratio wing perform best in most flight conditions. But when it comes to maximizing the lift from thermals, there are situations where a glider with a lower aspect ratio will perform better [1].

The subject of minimum rate of descent is well known and covered in aircraft design books and college textbooks. It is closely related to the minimum power required, maximum endurance, and best climb speed for a powered aircraft. Literature is available that addresses the minimum rate of descent for gliding flight [2] [3]. The derivations typically end with expressions in C_L and C_D. One published paper was found that expressed maximum endurance in terms of the aircraft design parameters [4]. No literature was found that expressed the minimum rate of descent for circling flight of a glider in terms of the aircraft design parameters. No literature was found that took the additional step of exploring the case of what happens when you fix the length of the wingspan.

The following mathematical derivations show how the design parameters affect the rate of descent for a new glider design with a fixed span length in straight and in circling flight. With the fixed-span design constraint, it will be shown that a lower aspect ratio can sometimes perform better. The results of these derivations are valuable to the designer of new gliders and can also be valuable to glider pilots and flight operations teams.

Targeting a Fixed Span Length

It is important to note how fixing the span length affects the design process. When a specific span length is targeted, the wing area becomes coupled with the aspect ratio by Equation 1.

$$A = \frac{b^2}{S}, \qquad b = constant$$

Equation 1

In addition, it is reasonable to assume that the designer will target a fixed spanwise distribution of lift. It will also be assumed that the weight of the aircraft will not change over the range of aspect ratios of interest (see Appendix A). The optimized structural weight of a glider wing primarily depends upon the spanwise bending moments. The bending moments will remain fixed if the span length and lift distribution are fixed.

With these assumptions, the wing loading also becomes coupled with the aspect ratio. Any decrease to the aspect ratio of a fixed span design will in turn increase the wing area and decrease the wing loading. This coupling of the design parameters alters the conclusions that can be drawn from examining the equations for the aircraft's minimum rate of descent.

The Aircraft Speed Polar

Deriving the rate of descent as a function of airspeed will allow us to identify the flight condition where the rate of descent is a minimum for a given design. The following derivations for straight, wings-level, gliding flight can be found in different forms in the available literature [2] [3]. It is included here for completeness and to form a basis for the derivations that show the relevant design parameters that affect the minimum rate of descent—in both level and circling flight.

Consider the forces acting upon a glider in straight flight.

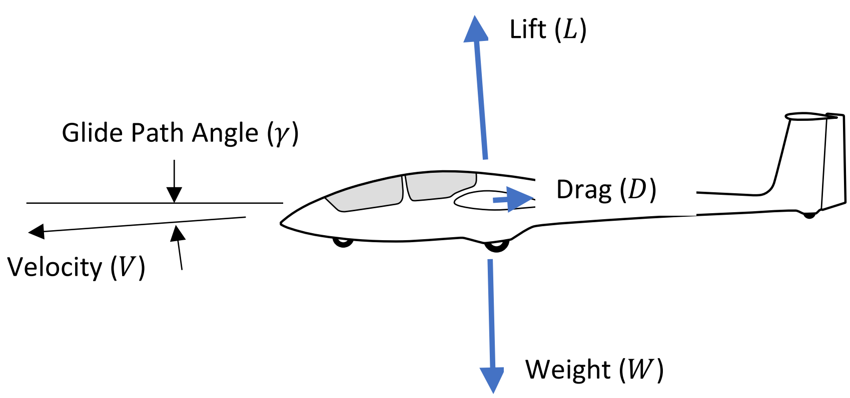

Figure 1. Free-body diagram for straight gliding flight.

For equilibrium, the following holds true.

$$L = W\cos\gamma\,, \qquad D = -W\sin\gamma$$

Equations 2 and 3

The rate of descent is then given by the following equation.

$$\dot{h} = V\sin\gamma = -V\frac{D}{W}$$

Equation 4

The aircraft drag is a function of airspeed and therefore must be expanded. The lift and drag for an aircraft are given by the following definitions.

$$L = qSC_L\,, \qquad D = qSC_D, \qquad q = \frac{1}{2}\rho V^2$$

Equations 5, 6, and 7

A commonly used and reasonable approximation for the drag coefficient is to define it as the following two-term quadratic [2].

$$C_D = C_{D_0} + \frac{C_L^2}{\pi A e}$$

Equation 8

Substituting Equations 5, and Equation 8 into Equation 4 gives us the following.

$$\begin{aligned}\dot{h} &= -V\frac{(qSC_D)}{W} \\ &= -\frac{1}{2}\rho V^3 \frac{S}{W}\left(C_{D_0} + \frac{C_L^2}{\pi A e}\right)\end{aligned}$$

Equation 9

Typically, the glide path angle for a glider is very small. And for small angles, we can use the following approximations.

$$\cos\gamma \approx 1, \qquad L \approx W, \qquad \therefore C_L \approx \frac{W}{qS}$$

Equations 10, 11, and 12

This will allow us to substitute for the lift coefficient in Equation 9, expand and simplify, to derive the speed polar for the aircraft.

$$\begin{aligned}\dot{h} &= -\frac{1}{2}\rho V^3 \frac{S}{W}\left[C_{D_0} + \left(\frac{W}{qS}\right)^2 \Big/ \pi A e\right] \\ &= -\frac{1}{2}\rho V^3 \frac{S}{W} C_{D_0} - \frac{1}{2}\rho V^3 \frac{S}{W}\left(\frac{W}{\frac{1}{2}\rho V^2 S}\right)^2 \Big/ \pi A e\end{aligned}$$

$$= -\frac{\rho S C_{D_0}}{2W} V^3 - \frac{2W}{\rho S \pi A e} V^{-1}$$

Equation 13

By way of example, consider a Standard Class glider with the specifications in Table 1. Figure 2 illustrates the speed polar for the aircraft flying at 5,000 feet altitude.

Weight	409 kg (900 lb)
Wing area	14.86 m^2 (160 ft^2)
Wingspan	15 m
Aspect ratio	15.1
Span efficiency	0.9
Minimum drag coefficient (C_{D_0})	0.01

Table 1. Example Standard Class glider specifications.

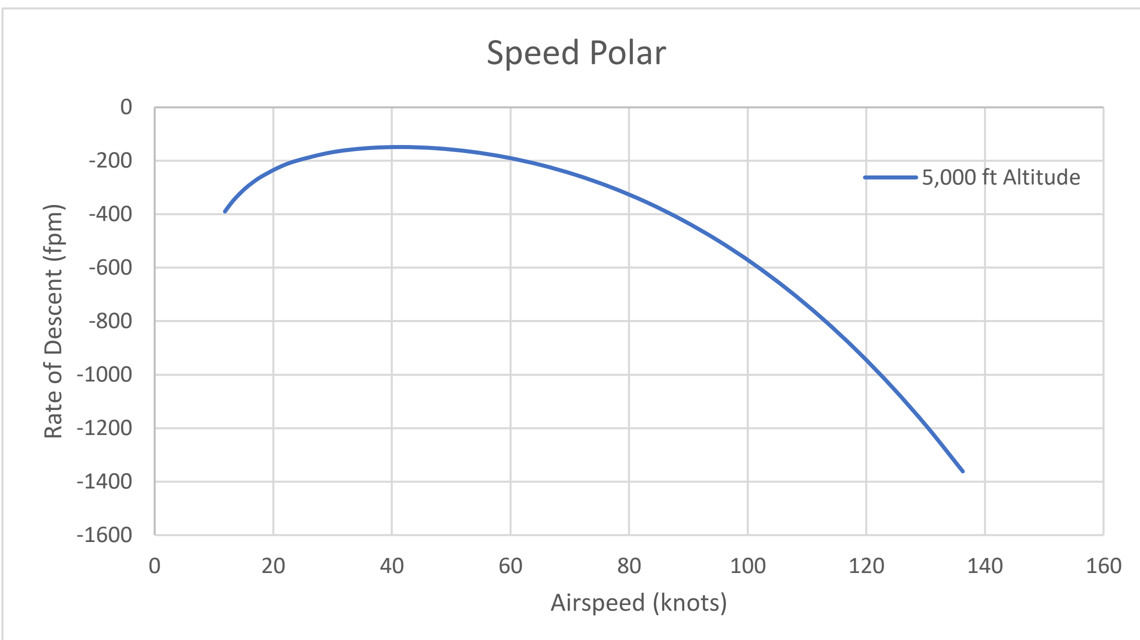

Figure 2. Speed polar for the example glider.

The Flight Condition for Minimum Rate of Descent

From the aircraft speed polar, we can observe that the minimum rate of descent occurs around 40 knots and where the slope of the speed polar curve is zero. It is therefore given that by taking the derivative of sink rate with respect to velocity and setting it to zero, we can determine the optimal flight condition for the minimum rate of descent.

Taking the derivative of Equation 13, factoring, and substituting in C_L from Equations 10, gives the following.

$$\begin{aligned}
\frac{\partial(\dot{h})}{\partial V} &= -\frac{3\rho S C_{D_0}}{2W} V^2 + \frac{2W}{\rho S \pi A e} V^{-2} \\
&= -\frac{3}{2}\rho V^2 \frac{S}{W}\left(C_{D_0} - \frac{4W^2}{3\rho^2 V^4 S^2 \pi A e}\right) \\
&= -3q\frac{S}{W}\left\{C_{D_0} - \frac{1}{3}\left[\frac{\left(W/\frac{1}{2}\rho V^2 S\right)^2}{\pi A e}\right]\right\} \\
&= -3q\frac{S}{W}\left[C_{D_0} - \frac{1}{3}\left(\frac{C_L^2}{\pi A e}\right)\right]
\end{aligned}$$

Equation 14

Setting the derivative to zero then identifies the lift coefficient for the minimum rate of descent in terms of the aircraft design parameters.

$$\begin{aligned}
&-3q\frac{S}{W}\left[C_{D_0} - \frac{1}{3}\left(\frac{C_L^2}{\pi A e}\right)\right] = 0 \\
&\rightarrow \quad C_{D_0} - \frac{1}{3}\left(\frac{C_L^2}{\pi A e}\right) = 0 \\
&\rightarrow \quad C_L = \sqrt{3C_{D_0}\pi A e}
\end{aligned}$$

Equation 15

Substituting Equation 15 into Equation 8, we can also identify the drag coefficient in terms of the aircraft design parameters.

$$\begin{aligned}
C_D &= C_{D_0} + \frac{\left(\sqrt{3C_{D_0}\pi A e}\right)^2}{\pi A e} \\
&= 4C_{D_0}
\end{aligned}$$

Equation 16

Design Parameters to Minimize the Rate of Descent in Straight Flight

We now have all the information we need to express the minimum rate of descent in terms of the aircraft design parameters. This will allow us to examine the equations to optimize the design and will show us which parameters are most important.

From the aerodynamic Equations 5, and the small angle approximation $L \approx W$, we can express V as follows.

$$V = \sqrt{\frac{2L}{\rho S C_L}} = \sqrt{\frac{2W}{\rho S C_L}}$$

Equation 17

From Equation 4, and substituting in Equation 17, we can arrive at a generalized equation for the rate of descent.

$$\begin{aligned}\dot{h} &= -V\frac{D}{L} = -V\frac{C_D}{C_L} \\ &= -\sqrt{\frac{2W}{\rho S C_L}}\frac{C_D}{C_L} \\ &= -\sqrt{\frac{2WC_D^2}{\rho S C_L^3}}\end{aligned}$$

Equation 18

Now substituting in the values of the lift and drag coefficients for the minimum rate of descent in terms of aircraft design parameters (Equation 15 and Equation 16), simplifying and grouping the constants, we obtain the equation for the minimum rate of descent in terms of the aircraft design parameters without any constraints.

$$\dot{h} = -\sqrt{\frac{2W\left(4C_{D_0}\right)^2}{\rho S\left(\sqrt{3C_{D_0}\pi Ae}\right)^3}}$$

$$= -\sqrt{\left(\frac{32}{(3\pi)^{3/2}\rho}\right)\frac{W}{S}\sqrt{\frac{C_{D_0}}{A^3 e^3}}}$$

Equation 19

From this we can see that to minimize the sink rate for straight and level flight, we need to lower the wing loading, lower C_{D_0}, increase the aspect ratio and span efficiency factor. This implies that if we target the lowest possible weight and C_{D_0}, and without a constraint on the span length, we could simply increase the aspect ratio to improve the performance.

Now let's see what happens when we add the constraint of a fixed-length span.

Substituting in for wing area from Equation 1 and grouping the constants, we have the final equation for the minimum rate of descent in terms of the design parameters for a fixed-span glider.

$$\dot{h} = -\sqrt{\left(\frac{32}{(3\pi)^{3/2}\rho}\right)\frac{W}{(b^2/A)}\sqrt{\frac{C_{D_0}}{A^3 e^3}}}$$

$$= -\sqrt{\left(\frac{32}{(3\pi)^{3/2}\rho b^2}\right)W\sqrt{\frac{C_{D_0}}{Ae^3}}}$$

Equation 20

We can see that it is still important to minimize weight, lower C_{D_0}, and increase the span efficiency factor. However, it is important to note that with the span fixed at a given length, the aspect ratio is now less important than it was. This is because any increase in the aspect ratio now implies an increase in the wing loading. There is a tradeoff between the two with the aspect ratio slightly winning out over the wing loading for straight flight.

Design Parameters to Minimize the Rate of Descent in Circling Flight

While it is true that minimizing the rate of descent for straight flight does help with loitering, gliders must circle to capture the most lift from a thermal. This is where things get a little different. In addition, the properties of a thermal determine what radius of turn, or bank angle, will result in the best rate of climb [1], and in turn affect the design decisions.

The following derivations will first focus on the performance of a glider in still air to express the minimum rate of descent in circling flight in terms of the aircraft design parameters. Then it will be shown how the results of the derivations can help make design decisions to maximize the rate of climb in thermal lift.

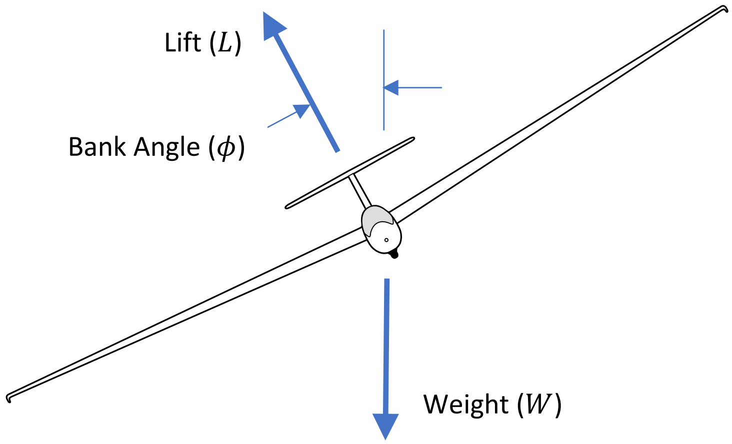

Figure 3. Free-body diagram for circling Flight.

Consider the forces acting upon a glider in circling flight. The aircraft bank angle gives rise to a centripetal force from the lift vector that turns the aircraft around in a circle. The total lift is divided between the centripetal acceleration and balancing the weight. Since we are focused on minimizing the rate of descent, it is reasonable to assume the glide path angles will remain small. Then for a steady-state turn, the following holds true.

$$W = L\cos\phi\,, \qquad \frac{WV^2}{Rg} = L\sin\phi$$

Equations 21 and 22

The velocity for circling flight can be expressed as follows.

$$V = \sqrt{\frac{2L}{\rho S C_L}} = \sqrt{\frac{2W}{\rho S C_L \cos\phi}}$$

Equation 23

And we can then derive a generalized equation for sink rate while circling.

$$\dot{h}_\phi = -V\frac{D}{L\cos\phi} = -V\frac{C_D}{C_L\cos\phi}$$
$$= -\sqrt{\frac{2W}{\rho SC_L\cos\phi}}\frac{C_D}{C_L\cos\phi}$$
$$= -\frac{1}{\cos^{3/2}\phi}\sqrt{\frac{2WC_D^2}{\rho SC_L{}^3}}$$
$$= \frac{\dot{h}_0}{\cos^{3/2}\phi}$$

Equation 24

Note that the sink rate for circling flight is simply a factor of the sink rate for level flight, and the magnitude of the sink rate in circling flight will always be higher. The cosine of an angle is always less than or equal to one.

Now we need to express Equation 24 in terms of the aircraft design parameters. From Equations 21, we can solve for $\sin\phi$ as follows.

$$\frac{WV^2}{Rg} = L\sin\phi = \frac{1}{2}\rho V^2 SC_L\sin\phi$$
$$\rightarrow \quad \sin\phi = \frac{2W}{\rho SC_L Rg}$$

Equation 25

Using a trigonometric identity ($\cos^2\phi + \sin^2\phi = 1$), $\cos\phi$ can then be expressed as follows.

$$\cos\phi = \sqrt{1-\left(\frac{2W}{\rho SC_L Rg}\right)^2}$$

Equation 26

And with that, the generalized equation for sink rate in circling flight can be expanded.

$$\dot{h}_\phi = \frac{\dot{h}_0}{\left[1-\left(\frac{2W}{\rho SC_L Rg}\right)^2\right]^{3/4}}$$

Equation 27

Substituting the C_L for minimum rate of descent (Equation 15) and grouping the constants gives us the equation for the minimum rate of descent in circling flight in terms of the aircraft design parameters without constraints.

$$\dot{h}_\phi = \frac{\dot{h}_0}{\left[1-\left(\frac{2W}{\rho S\sqrt{3C_{D_0}\pi AeRg}}\right)^2\right]^{3/4}} = \frac{\dot{h}_0}{\left[1-\left(\frac{4}{3\pi(\rho g)^2}\right)\frac{W^2}{S^2 C_{D_0} AeR^2}\right]^{3/4}}$$

Equation 28

Adding the constraint of a fixed-length span, we substitute in for the wing area from Equation 1 and group the constants to give the following equation.

$$\dot{h}_\phi = \frac{\dot{h}_0}{\left[1-\left(\frac{4}{3\pi(\rho g)^2}\right)\frac{W^2}{(b^2/A)^2 C_{D_0} AeR^2}\right]^{3/4}} = \frac{\dot{h}_0}{\left[1-\left(\frac{4}{3\pi(\rho g)^2 b^4}\right)\frac{W^2 A}{C_{D_0} eR^2}\right]^{3/4}}$$

Equation 29

Lastly, we insert the equation for h_0 (Equation 20) to produce the final equation for the minimum rate of descent in circling flight in terms of the aircraft design parameters for a fixed-span glider.

$$\dot{h}_\phi = \frac{-\sqrt{\left(\frac{32}{(3\pi)^{3/2}\rho b^2}\right)W\sqrt{\frac{C_{D_0}}{Ae^3}}}}{\left[1-\left(\frac{4}{3\pi(\rho g)^2 b^4}\right)\frac{W^2 A}{C_{D_0} eR^2}\right]^{3/4}}$$

Equation 30

We can see that it is still important to minimize the aircraft weight and C_{D_0}. It is also important to increase the span efficiency factor. But now, we have the aspect ratio in offsetting terms.

Figure 4 shows visually how the wing design parameters affect the minimum rate of descent for the example 15-meter glider given in Table 1. For the smaller radius turns, it is better to have a lower wing loading, and consequently a lower bank angle, than it is to have a higher aspect ratio wing. In those conditions, the lower wing loading is more important.

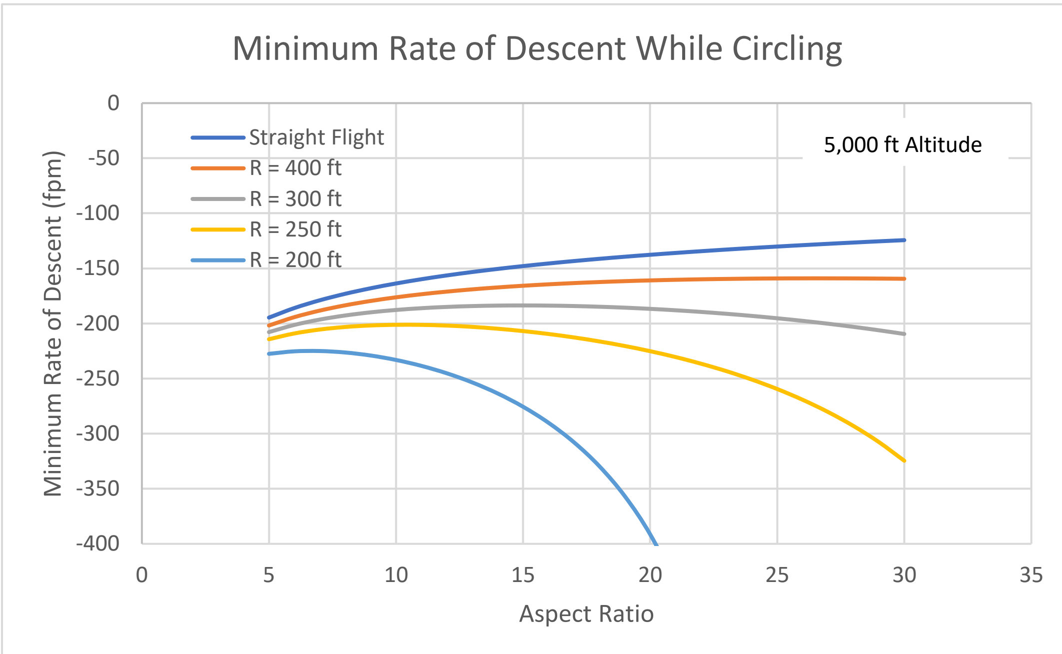

Figure 4. Minimum rate of descent for the example glider while circling.

The Effects of Thermal Lift

The whole purpose of trying to minimize the rate of descent while circling is to maximize the altitude gain while circling in a thermal. Interestingly, the structure of the thermal lift will affect the design decisions.

Pivoting the data generated from Equation 30 and plotting the minimum rate of descent versus the turn radius, we can now add the lift profile of a thermal. The standard British thermal is defined as being a circular thermal with a cross section having 4.2 knots of lift at the core and decreasing parabolically to zero at 1,000 feet [3]. This thermal profile is used by the British Gliding Association (BGA) for handicapping purposes. Figure 5 shows the results for our example glider over a range of aspect ratios. For the standard British thermal, our example glider would perform best with an aspect ratio of approximately 17.

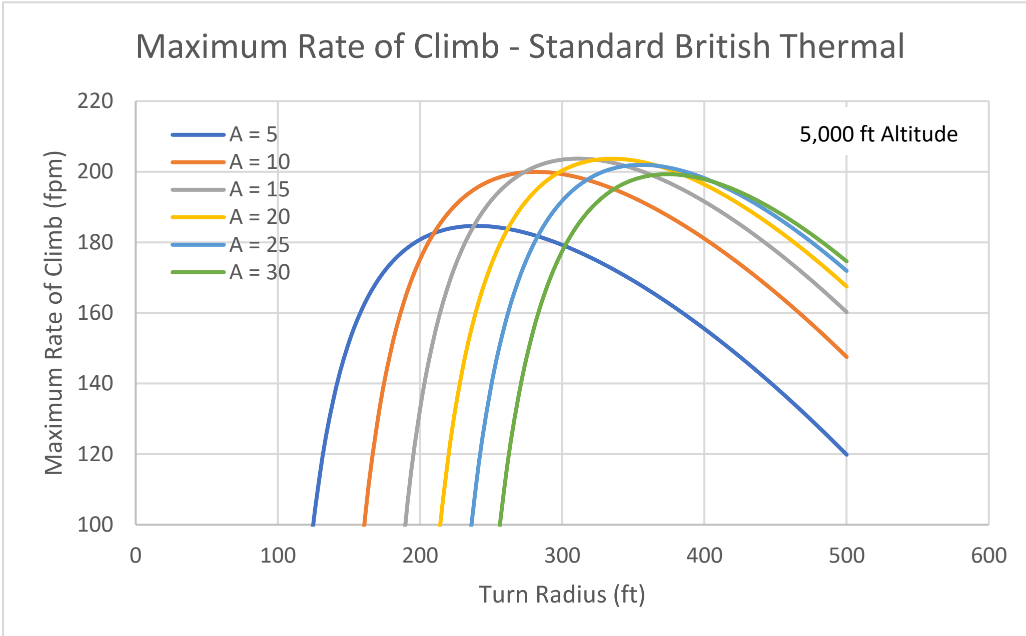

Figure 5. Maximum rate of climb for the example glider in a standard British thermal.

Not all thermals are the same. If the thermal was stronger and narrower, our conclusions would be different. Consider a thermal with an 8-knot core and decreasing parabolically to zero at 800 feet. Figure 6 shows that our example glider would perform best with an aspect ratio of approximately 8.

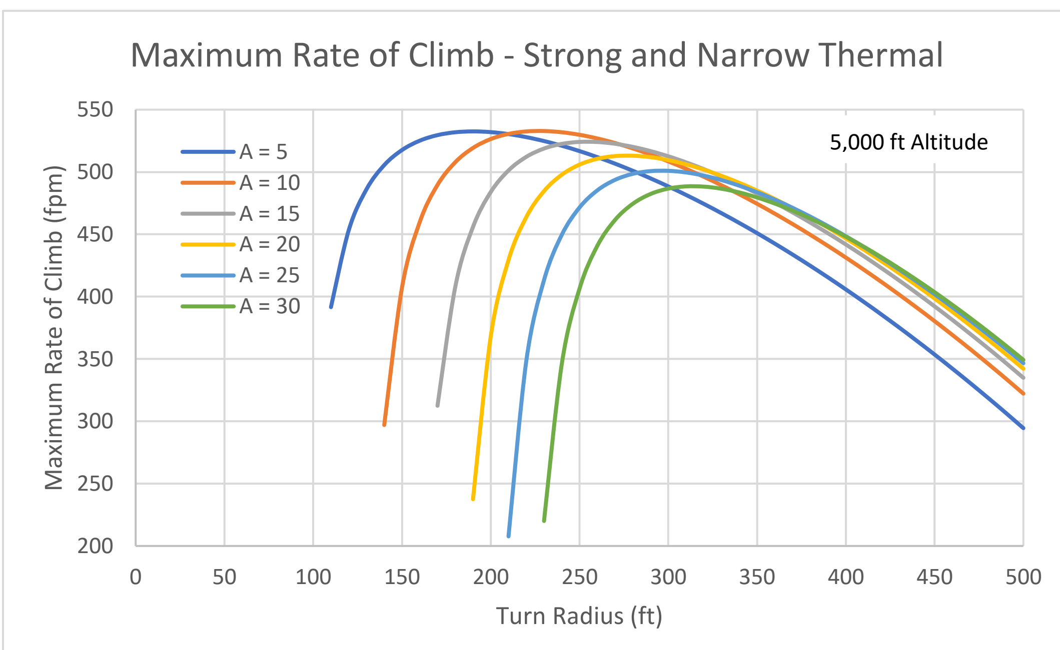

Figure 6. Maximum rate of climb for the example glider in a strong and narrow thermal.

Conclusions

By deriving the mathematical equations that express the minimum rate of descent for a glider with a fixed span length, it has been shown that a lower aspect ratio wing can perform better in circling flight. The results for circling flight were then applied to the lift profile of two different thermals. It was shown that the optimum aspect ratio for circling flight depends on the strength and width of the thermal.

It was also shown that a lighter aircraft with a higher span efficiency factor and a lower minimum drag coefficient will perform better in all flight conditions.

The final design decision for a glider aspect ratio then becomes a compromise. The designer must evaluate the ratio of time the aircraft will be gliding straight versus the time it will be circling in lift. The designer must also decide on the structure of lift the glider will most likely encounter.

If possible, and in the case of a competition 15-meter Standard Class glider, it would be best to have two or three sets of wings. The pilot and flight operations team could then decide which set of wings to use for the day. The lower aspect ratio wing will provide lower wing loading and improved climb performance in strong and narrow thermals. It would be the best choice when strong or narrow thermals are abundant. The higher aspect ratio wing will perform better in light and more uniform lift conditions. Also, the higher aspect ratio wing would make it possible to increase the wing loading when needed without increasing the total aircraft weight. This could be done before resorting to ballast.

Appendix A

In some of the most recognized studies on spanwise lift distributions, the wing bending moments are held constant while the span length and loading is optimized [5] [6]. The implied assumption in these works is that if you hold constant the integrated bending moments, or simply the bending moment at the root of the wing, the weight of the wing will not change. Although it is intuitive that the loading on a wing will dictate the required structural weight, unfortunately the implied assumptions from prior work cannot be used as proof. It would be instructive then to examine what happens in practice.

Richard Shevell in his well-known book, *Fundamentals of Flight*, shows that the weight of a wing per unit square foot can be estimated with relative accuracy by the following equation where A_1 and B_1 are constants determined from empirical data [7].

$$W_w/ft^2 = A_1 + B_1 I_w$$

Equation 31

The term I_w is called the "wing weight index" and is proportional to the skin and spar cap compression and tension loads due to the bending moments at the root of the wing. For a glider, the "wing weight index" is given by the following.

$$I_w = \frac{N_{ULT} b^3 W (1 + 2\sigma)}{(t/c)_{AVG} \cos^2 \Lambda S^2 (1 + \sigma)} \times 10^{-6}$$

Equation 32

The total estimated weight of a wing is obtained by multiplying Equation 31 by the wing area.

$$\begin{aligned} W_w &= S(A_1 + B_1 I_w) \\ &= A_1 S + B_1 S I_w \end{aligned}$$

Equation 33

The term $A_1 S$ represents all the miscellaneous components that do not contribute to supporting the structural loads (i.e. control systems, paint, etc.). For a glider with a fixed span length, a lower aspect ratio wing would have a marginal increase in weight from this term.

The term $B_1 S I_w$ represents the structural weight of the wing. It can be assumed that the ultimate load factor, wing sweep, and taper ratio would be held constant by the designer of a fixed span glider. Simplifying the terms and grouping the constants, the term "SI_w" can be written as follows.

$$SI_w = S\left[\frac{N_{ULT} b^3 W (1 + 2\sigma)}{(t/c)_{AVG} \cos^2 \Lambda S^2 (1 + \sigma)} \times 10^{-6}\right]$$

$$
\begin{aligned}
&= \left[\frac{N_{ULT} b^3 W(1+2\sigma)}{(t/c)_{AVG} \cos^2\Lambda\, S(1+\sigma)} \times 10^{-6}\right] \\
&= \left[\frac{N_{ULT} W(1+2\sigma)}{\cos^2\Lambda\, S(1+\sigma)} \times 10^{-6}\right]\left[\frac{b^3}{(t/c)_{AVG} S}\right] \\
&= B_2\left[\frac{b^3}{(t/c)_{AVG} S}\right], \qquad where\ B_2 = \left[\frac{N_{ULT} W(1+2\sigma)}{\cos^2\Lambda\, S(1+\sigma)} \times 10^{-6}\right]
\end{aligned}
$$

Equation 34

Recognizing that $S = bc_{AVG}$ and making the further assumption that the section thickness ratio t/c be held constant down the span, we can further simplify the equation.

$$
\begin{aligned}
SI_w &= B_2\left[\frac{b^3}{(t/c)_{AVG}(bc_{AVG})}\right] \\
&= B_2 \frac{b^2}{t_{AVG}}
\end{aligned}
$$

Equation 35

For a glider with a fixed span length ($b = constant$), the structural weight of the wing is then inversely proportional to the wing thickness. This would indicate that a lower aspect ratio wing would have a lower structural weight. The lower aspect ratio wing has a lower aerodynamic load per unit area and a taller main spar. However, it is likely the designer would choose a thinner airfoil section for the lower aspect ratio wing to help offset the increased skin friction drag. It is reasonable to assume that a compromise of all the factors would result in the structural weight being only slightly lower and comparable to the marginal increase in weight from the non-structural items (paint, etc.). The lower aspect ratio wing would weigh approximately the same as the higher aspect ratio wing.

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